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A New Solution to the Random Assignment Problem

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A random assignment is *ordinally efficient* if it is not stochastically dominated with respect to individual preferences over sure objects. Ordinal efficiency implies (is implied by) ex post (ex ante) efficiency. A simple algorithm characterizes ordinally efficient assignments: our solution, probabilistic serial (PS), is a central element within their set. Random priority (RP) orders agents from the uniform distribution, then lets them choose successively their best remaining object. RP is ex post, but not always ordinally, efficient. PS is envy-free, RP is not; RP is strategy-proof, PS is not. Ordinal efficiency, Strategyproofness, and equal treatment of equals are incompatible. *Journal of Economic Literature* Classification Numbers: C78, D61, D63.

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1. TWO PREVIOUS SOLUTIONS TO THE RANDOM ASSIGNMENT PROBLEM

The *assignment problem* is the allocation problem where n objects are to be allocated among n agents, and each agent is to receive exactly one object. Examples include the assignment of jobs to workers, of rooms to housemates, of time slots to users of a common machine, and so on (see Roth and Sotomayor [14], Abdulkadiroglu and Sönmez [1]).

Using a lottery is one of the oldest tricks (going further back than the *Bible*; see Hofstee [9]) to restore fairness in such problems.[1] Suppose we

[1] Another standard trick is to use monetary compensations; see Leonard [12], Demange [5]. Here we assume that money is not available.

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must allocate a single (desirable) object among several agents: tossing a fair die is obviously the uniquely optimal method.[2] The simplest extension of this method to the case of n heterogeneous objects and n agents is the familiar mechanism that we call *random priority*[3]: draw at random an ordering of the agents from the uniform distribution and then let them successively choose an object in that order (so the first agent in the ordering gets first pick and so on). This method has been around for a long time, although only two papers in the economic literature discuss it: Zhou [19] and Abdulkadiroglu and Sönmez [1].

From the point of view of mechanism design, random priority is fair (at least in the sense of equal treatment of equals) and incentive compatible (in the sense of strategyproofness), but it is not efficient when the agents are endowed with Von Neumann–Morgenstern preferences over random allocations (lotteries over objects), see Zhou [19].[4]

A second, more subtle, solution to the random assignment problem was proposed by Hylland and Zeckhauser [10]. It adapts the competitive equilibrium with equal incomes solution for the fair division of unproduced commodities (Varian [18], Thomson and Varian [17]) to the random assignment model: a VNM utility function over random allocations of (indivisible) objects is viewed as a linear utility over vectors of "shares" of these objects, where the share of an object is the probability of receiving it in the eventual assignment. This solution, denoted CEEI, is fair in the sense of no envy (a criterion that the RP assignment does not always meet; see Proposition 1 in Section 7). It is also efficient with respect to the profile of VNM utility functions, yet it is not strategyproof. In fact, Zhou [19] (proving a conjecture formulated in Gale [7]) shows that there does not exist any strategyproof mechanism eliciting individual VNM utility functions and achieving both efficiency (Pareto optimality w.r.t. these utility functions) and equity (in the very weak sense of equal treatment of equals).

The RP assignment is appealing on two accounts. First it is *ex post efficient*, that is to say every deterministic assignment that is selected with positive probability is Pareto optimal. This is a weaker property than *ex ante efficiency*, namely efficiency with respect to the profile of VNM utility functions, as discussed above. Second, the RP assignment can be computed

[2] For a formal statement see comment b in Section 9.

[3] Also called random serial dictatorship by Abdulkadiroglu and Sönmez [1].

[4] An example follows, with three agents, three objects and the VNM utilities: $u_1(a)=1$, $u_1(b)=0.8$, $u_1(c)=0$, for $i=2, 3$: $u_i(a)=1$, $u_i(b)=0.2$, $u_i(c)=0$. The random priority assignment gives a 1/3 chance of every object to each agent (because their preferences over sure objects coincide) and a profile of expected utilities (0.6, 0.4, 0.4). But assigning object b to agent 1 for sure and objects a, c randomly between agents 2, 3 yields the expected utilities (0.8, 0.5, 0.5).

or implemented from the profile of preferences over *sure* objects. For any given ordering of the agents, we compute the best available object for each agent in turn, and this does not require knowledge of the preferences over lotteries (random objects). Contrast this simplicity with the computation of the CEEI solution, which requires us to elicit the profile of full-fledged VNM utility functions and to solve a difficult fixed-point problem (of which the solution may not be unique). We submit that the simplicity of information gathering and implementation of the RP assignment makes the corresponding mechanism more appealing than the CEEI one.

We call a mechanism that elicits only individual preferences over sure objects *ordinal* (whereas a cardinal mechanism collects a full-fledged VNM utility function from every agent). The restriction to ordinal mechanisms is the central assumption in this paper.[5] It can be justified by the limited rationality of the agents participating in the mechanism. There is convincing experimental evidence that the representation of preferences over uncertain outcomes by VNM utility functions is inadequate (see, e.g., Kagel and Roth [11]). One interpretation of this literature is that the formulation of rational preferences over a given set of lotteries is a complex process that most agents do not engage into if they can avoid it. An ordinal mechanism allows the participants to formulate only this part of their preferences that does not require to think about the choice over lotteries. It is genuinely simpler to implement an ordinal mechanism than a cardinal one.

We introduce a new notion of efficiency for the random assignment problem that we call ordinal efficiency (O-efficiency). O-efficiency is a consequence of ex ante efficiency, and it implies ex post efficiency. It relies on preferences over sure objects only; thus it is attainable by an ordinal mechanism. We show that the RP assignment may be ordinally inefficient. We propose a canonical O-efficient assignment that we call the probabilistic serial assignment.

2. ORDINAL EFFICIENCY AND THE NEW SOLUTION

Preferences over deterministic objects induce a *partial ordering* over random allocations (i.e., lotteries over objects), namely the (first order) stochastic dominance relation. Our notion of ordinal efficiency (Definition 1) comes from the Pareto (partial) ordering induced by the stochastic dominance relations of individual agents.

[5] Gibbard [8] considers ordinal mechanisms in the context of voting with lotteries (random choice of a pure public good). See also Ehlers *et al.* [6] in the context of a fair division problem different from ours.

Ordinal efficiency is a stronger requirement than ex post efficiency. We give a simple example with four agents,[6] where the RP assignment is ordinally inefficient. The preferences are as follows:

$$\begin{aligned} &\text{agents 1, 2:} \quad a \succ b \succ c \succ d \\ &\text{agents 3, 4:} \quad b \succ a \succ d \succ c. \end{aligned} \tag{1}$$

The RP assignment gives to agents 1, 2 a positive probability of receiving b (e.g., for the ordering 1234) and to agents 3, 4 a positive probability of receiving a. More precisely the RP assignment is as follows, where rows marked 1–4 are agents and columns a–d are objects.

$$\begin{array}{ccccc} & a & b & c & d \\ 1 & 5/12 & 1/12 & 5/12 & 1/12 \\ 2 & 5/12 & 1/12 & 5/12 & 1/12 \\ 3 & 1/12 & 5/12 & 1/12 & 5/12 \\ 4 & 1/12 & 5/12 & 1/12 & 5/12 \end{array} \tag{2}$$

Yet the following assignment is preferred by *every* agent to (2), irrespective of their VNM utility functions compatible with the preferences (1):

$$\begin{array}{ccccc} & a & b & c & d \\ 1 & 1/2 & 0 & 1/2 & 0 \\ 2 & 1/2 & 0 & 1/2 & 0 \\ 3 & 0 & 1/2 & 0 & 1/2 \\ 4 & 0 & 1/2 & 0 & 1/2 \end{array} \tag{3}$$

On the other hand, ordinal efficiency is a weaker requirement than ex ante efficiency, as can be seen easily by considering a profile of identical preferences over sure objects (just like in footnote 4 above) where every feasible assignment is ordinally efficient.

Our first result (Theorem 1 in Section 5) characterizes the entire set of ordinally efficient assignments by a natural constructive algorithm. Think of each object as 1 unit of an infinitely divisible commodity (a different commodity for each object), where "agent i receives 0.4 units of commodity a" means that agent i gets a with probability 0.4. Each agent now is given an exogeneous *eating speed function*, specifying a rate of instant consumption for each time t between 0 and 1, and such that the integral of each function is 1. Given a profile of preferences (over sure objects), the algorithm works as follows: each agent eats from his or her best *available* object at the

[6] With two or three agents, ordinal and ex post efficiency coincide; see Lemma 2 in Section 4.

given speed, where an object is available at time t if and only if less than one unit has been eaten away up to time t by all agents.

Our new solution, probabilistic serial (PS), obtains when the eating speed functions are identical across agents (hence it can be taken to be the unit eating speed between $t=0$ and $t=1$). The PS assignment is a central point in the set of ordinally efficient assignments; if we choose the eating speeds independent of the preference profiles, the PS assignment is the only equitable one (Lemma 4 in Section 6). The RP assignment is similarly a central point within the set of ex post efficient assignments (see Lemma 1 in Section 4).

In example (1) above, the PS algorithm does the right thing, namely it selects assignment (3): indeed agents 1 and 2 start by eating object a while agents 3, 4 start by eating object b. As the common speed is one, at time $t=0.5$ both a and b are entirely consumed, and any inefficient allocation of a to agents 3, 4 or of b to agents 1, 2 is avoided. The same applies to the allocation of objects c, d between $t=0.5$ and $t=1$.

We systematically compare the RP and PS mechanisms for their efficiency, fairness, and incentives properties. We find that PS fares better on efficiency and fairness, yet RP has better incentives properties. On the former, the PS assignment is always ordinally efficient (Theorem 1) and may stochastically dominate the RP assignment for every agent; moreover the PS assignment is envy-free for any profile of VNM utilities over lotteries compatible with the given preferences over sure objects, whereas the RP assignment only meets a weaker version of this criterion (Proposition 1 in Section 7). On the latter, the RP mechanism is strategyproof for any profile of VNM utilities over lotteries, whereas the PS mechanism is only strategyproof in a weaker sense (Proposition 1).

Our third and last contribution is an impossibility result similar (but not logically related) to Zhou's theorem. We show that with four agents or more, there is no ordinal mechanism meeting simultaneously ordinal efficiency, strategyproofness and equal treatment of equals (Theorem 2 in Section 8). This does not imply Zhou's theorem because the class of mechanisms that we consider is smaller than his, yet it carries the same negative implications: even the huge conceptual simplification of only looking at ordinal mechanisms is not enough to ensure the compatibility of the three familiar requirements, efficiency, fairness, and incentive compatibility.

In Section 3 we discuss the literature related to our results. Section 4 defines the model, introduces the concept of ordinal efficiency, and compares it to ex ante and to ex post efficiency. In Section 5 we characterize the set of ordinally efficient assignments by means of the eating algorithms mentioned above. The probabilistic serial assignment is defined and compared with the RP one from the efficiency angle in Section 6. In Section 7 we compare RP and PS from the fairness (envy-freeness) and incentive compatibility (strategyproofness) angles. Section 8 offers two characterization results, respectively, of the PS and RP mechanisms, in the case of three agents; it also states the impossibility result parallel to Zhou's. The final Section 9 discusses a number of variants of our model, to which our results are easy to adapt. We consider successively a different number of objects and agents, the possibility of "objective" indifferences (some objects are viewed as identical by all agents), and the possibility of opting out.

All proofs are provided in the Appendix.

3. RELATED LITERATURE

The literature on the random assignment problem is very small and we have already discussed two of its oldest papers: Hylland and Zeckhauser [10] who propose to adopt the competitive equilibrium with equal incomes, and Zhou [19], who proves an important impossibility result.

More recently Abdulkadiroglu and Sönmez [1] show that the random priority assignment obtains as the top trading cycle outcome due to Shapley and Scarf [16] with random initial endowments.

Crès and Moulin [4] introduce the PS mechanism in a simple model of assignment with common ranking of the objects and where agents can opt out. In that model, the PS assignment stochastically dominates the RP one. In the same model, Bogomolnaia and Moulin [3] introduce the concept of ordinal efficiency and give two characterizations of the PS mechanism, briefly discussed in comment *d* of Section 9.

Two recent papers have improved an earlier version of this work. Abdulkadiroglu and Sönmez [2] uncover some surprising features of ex post efficiency (thus correcting an erroneous statement in the first version of this paper) and offer an alternative characterization of ordinal efficiency; see the discussion after Definition 1, Section 4. McLennan [13] proves an important link between ordinal efficiency and ex ante efficiency: his result is described in Remark 2, Section 4.

Finally, Ehlers *et al.* [6] apply the same notion of ordinal efficiency in the probabilistic version of the problem of fair division with single-peaked preferences. In their model, ordinal and ex post efficiency coincide.

4. EX POST, EX ANTE AND ORDINAL EFFICIENCY

First we define the random assignment problem. Throughout the paper, the sets N of agents and A of objects are fixed, both finite and of cardinality n.

A *deterministic assignment* is a one-to-one mapping from N into A; it will be represented as a permutation matrix (a $n \times n$ matrix with entries 0 or 1 and exactly one nonzero entry per row and one per column) and denoted Π. We identify rows with agents and columns with objects. We denote by $\mathscr{D}$ the set of deterministic assignments.

A *random allocation* is a probability distribution over A; their set is denoted $\mathscr{L}(A)$.

A *random assignment* is a probability distribution over deterministic assignments. The corresponding convex combination of permutation matrices describes the probabilities that a given agent receives a given good:

$$P = [p_{ia}]_{i \in N,\, a \in A} \qquad \text{where} \quad P = \sum_{\Pi \in \mathscr{D}} \lambda_\Pi \cdot \Pi \quad \text{and} \quad \lambda_\Pi \geqslant 0, \quad \sum_{\Pi} \lambda_\Pi = 1.$$

The matrix P is bistochastic, and its ith row is denoted P_i. It is agent i's random allocation:

$$\text{for all} \quad i \in N, \quad a \in A; \qquad p_{ia} \geqslant 0, \quad \sum_{j \in N} p_{ja} = \sum_{b \in A} p_{ib} = 1. \tag{4}$$

By the classical Birkhoff–Von Neumann theorem, every bistochastic matrix obtains as a (in general, not unique) convex combination of permutation matrices: hence every such matrix corresponds to some random assignment(s).

Two probability distributions over $\mathscr{D}$ resulting in the same bistochastic matrix will not be distinguished (they yield the same welfare level to every agent). Therefore we identify a random assignment and its bistochastic matrix P. We denote by $\mathscr{R}$ the set of bistochastic matrices.

Each agent i is endowed with *strict* preferences $\succ_i$ over A. We denote this domain of preferences by $\mathscr{A}$. We note that ruling out indifferences is not an innocuous assumption: our results do not extend straightforwardly to allow for indifferences among objects, except in the case of objective indifferences. More on this in Section 9, comments b and c.

A Von Neumann–Morgenstern utility function (VNM utility) u_i is a real valued mapping on A: the corresponding preferences over $\mathscr{L}(A)$ obtain by comparing expected utilities $u_i \cdot P_i = \sum_a u_i(a) \cdot p_{ia}$. We say that u_i is *compatible* with the (strict) preference $\succ_i$ when $u_i(a) \succ u_i(b)$ *iff* $a \succ_i b$.

For a deterministic assignment Π, there is only one notion of efficiency, and the efficient subset of $\mathscr{D}$ is entirely described by the *priority assignments* that we now define.

An ordering of N is a one-to-one mapping σ from $\{1, 2, ..., n\}$ into N; we denote by θ the set of such orderings. Given the ordering σ and the preference profile $\succ$, the corresponding *priority assignment* is denoted *Prio*$(\sigma, \succ)$ and defined as follows: agent $\sigma(1)$ receives his or her best object a_1 in A (according to $\succ_{\sigma(1)}$); agent $\sigma(2)$ receives his or her best object a_2 in $A \backslash \{a_1\}$; agent $\sigma(t)$ receives his or her best object a_t in $A \backslash \{a_1, ..., a_{t-1}\}$.

LEMMA 1. *Fix a profile of preferences $\succ$ in $\mathscr{A}^N$ and a deterministic assignment Π. The three following statements are equivalent*

(i) *Π is Pareto optimal in $\mathscr{D}$ at $\succ$,*

(ii) *for any profile of VNM utilities $(u_i, i \in N)$ compatible with the profile $\succ$, Π is Pareto optimal in $\mathscr{R}$ at u,*

(iii) *there exists an ordering σ of N such that $\Pi = Prio(\sigma, \succ)$.*

We omit the easy proof.

DEFINITION 1. Given a random assignment P, $P \in \mathscr{R}$, a profile of preferences $\succ$ in $\mathscr{A}^N$, and a profile of VNM utilities u, we define:

(i) P is *ex ante* efficient at u *iff* P is Pareto optimal in $\mathscr{R}$ at u

(ii) P is *ex post* efficient at $\succ$ *iff* it can be represented as a probability distribution over efficient deterministic assignments. That is to say, P takes the form

$$P = \sum_{\sigma \in \theta} \mu_\sigma \, Prio(\sigma, \succ) \qquad \text{for some convex system of weights } \mu_\sigma. \tag{5}$$

In view of (5), a natural central point within the set of ex post efficient assignments simply takes the uniform system of weights. This is the *random priority* assignment

$$RP(\succ) = \frac{1}{n!} \sum_{\sigma \in \theta} Prio(\sigma, \succ). \tag{6}$$

Remark 1. Abdulkadiroglu and Sönmez [2] observe that some ex post efficient random assignments P can also be represented as a lottery over *inefficient* deterministic assignments (they show this to be possible for $n \geqslant 4$). Therefore ex post efficiency is a subtle concept because of the possibility of multiple representations of a doubly stochastic matrix as a lottery over deterministic assignments.

We are now ready to define the concept of ordinal efficiency. A given preference ordering $\succ_i$ on A induces a partial ordering of the set $\mathscr{L}(A)$ of random allocations that we call the *stochastic dominance* relation associated with $\succ_i$ and denote $sd(\succ_i)$. Upon enumerating A from best to worst according to $\succ_i$: $a_1 \succ_i a_2 \succ_i a_3 \succ_i \cdots \succ_i a_n$, we define

$$\text{for all } P_i, \quad Q_i \in \mathscr{L}(A) : P_i \, sd(\succ_i) \, Q_i \overset{\text{def}}{\Leftrightarrow} \left\{ \sum_{k=1}^{t} p_{ia_k} \geqslant \sum_{k=1}^{t} q_{ia_k}, \text{ for } t = 1, ..., n \right\}.$$

Note that the relation $sd(\succ_i)$ is reflexive ($P_i\, sd(\succ_i)\, P_i$) whereas $\succ_i$ is not. Clearly the statement $P_i\, sd(\succ_i)\, Q_i$ is equivalent to the property that $u_i \cdot P_i \geqslant u_i \cdot Q_i$ for any VNM-utility function u_i on A compatible with $\succ_i$. If, moreover, $P_i \neq Q_i$, then we have $u_i \cdot P_i > u_i \cdot Q_i$ for any such utility function.

DEFINITION 2. Given the preference profile ($\succ_i$, $i \in N$), we say that the random assignment P, $P \in \mathscr{R}$, is *stochastically dominated* by another random assignment Q, $Q \in \mathscr{R}$, if we have:

$$\{Q_i\, sd(\succ_i)\, P_i \text{ for all } i\} \qquad \text{and} \qquad Q \neq P.$$

We say that P is *ordinally efficient* (O-efficient) if it is not stochastically dominated.

We compare ordinal efficiency with the two other notions of efficiency introduced in Definition 2. The first observation is that O-efficiency, like ex post efficiency and unlike ex ante efficiency, only depends upon the profile of individual preferences over sure objects, namely upon the preferences over A.

LEMMA 2. *Fix a random assignment P, $P \in \mathscr{R}$, a preference profile $\succ$ in $\mathscr{A}^N$, and a profile u of VNM utilities compatible with $\succ$ (that is, u_i is compatible with $\succ_i$ for all i).*

(i) *If P is ex ante efficient at u, then it is ordinally efficient at $\succ$; the converse statement holds for $n = 2$ but may fail for $n \geqslant 3$.*

(ii) *If P is ordinally efficient at $\succ$, then it is ex post efficient at $\succ$; the converse statement holds for $n \leqslant 3$ but may fail for $n \geqslant 4$.*

Remark 2. Lemma 2 implies the following: given a profile $\succ$ of preferences over sure objects and a random assignment P, P is ordinally efficient *if* there exists a profile u of VNM utilities compatible with $\succ$ and such that P is ex ante efficient at u. McLennan [13] proves the converse statement: if P is ordinally efficient then such a profile of VNM utilities exists.

5. ORDINAL EFFICIENCY AND THE SIMULTANEOUS EATING ALGORITHM

We give two characterizations of ordinal efficiency. The first one determines whether a given element P in $\mathscr{R}$ is ordinally efficient by checking the acyclicity of a certain relation constructed from P and the preference profile. The second one is a family of algorithms from which we can construct the whole subset of ordinally efficient assignments in $\mathscr{R}$.

Given a preference profile $\succ$ and a random assignment P, we define a binary relation in A as follows:

$$\text{for all} \quad a, b \in A : a\tau(P, \succ)b \Leftrightarrow \{\text{there exists } i \in N : a \succ_i b \text{ and } p_{ib} > 0\}. \tag{7}$$

LEMMA 3. *The random assignment P, $P \in \mathcal{R}$, is ordinally efficient at profile $\succ$ if and only if the relation $\tau(P, \succ)$ is acyclic.*

The second characterization result relies on a family of intuitive algorithms. Think of each object as an infinitely divisible commodity of which one unit must be distributed between the n agents. A quantity p_{ia} of good a allocated to agent i is implemented by giving object a to agent i with probability p_{ia}.[7]

Each one of our *simultaneous eating* algorithms relies on a set of n *eating speed* functions ω_i, $i = 1, ..., n$. Thus $\omega_i(t)$ is the speed at which agent i is allowed to eat at time t. The speed $\omega_i(t)$ is nonnegative and the total amount that agent i will eat between $t = 0$ and $t = 1$ (the end time of the algorithm) is one:

$$\int_0^1 \omega_i(t)\, dt = 1.$$

Given the profile of eating speeds $\omega = (\omega_i)_{i \in N}$ and the profile $\succ$ of preferences, the algorithm lets each agent i eat his or her best available good at the prespecified speeds: if at time t the objects $a, b, c...$ have been entirely eaten away (one unit of each has been distributed) and the objects $x, y, z, ...$ have not, agent i eats from his or her best object among $x, y, z, ...$ at speed $\omega_i(t)$.

For instance, consider the profile of uniform eating speeds $\omega_i(t) = 1$ for all t, $0 \leqslant t \leqslant 1$, all $i = 1, 2, 3, 4$ in Example (1), Section 2. From $t = 0$ until $t = 0.5$ agents 1 and 2 eat good a whereas agents 3, 4 eat good b; both goods are exhausted at $t = 0.5$; hence from $t = 0.5$ until $t = 1$ agents 1 and 2 eat good c whereas agents 3 and 4 eat good d. The resulting outcome is precisely the random assignment (3). We turn to the formal definition of the simultaneous allocation algorithms.

Denote by W the domain of eating speed functions:

$$\omega_i \in W \quad \textit{iff } \omega_i \text{ is a measurable function,} \quad \omega_i\colon [0, 1] \to \mathbf{R}_+, \quad \int_0^1 \omega_i(t)\, dt = 1.$$

[7] Recall that the solution proposed by Hylland and Zeckhauser [10] uses this interpretation to let agents buy competitively some shares in the various objects.

We use the following notation: whenever $a \in B$, let $M(a, b) \overset{\text{def}}{=} \{i \in N \mid a \succ_i b \; \forall b \in B,\; b \neq a\}$, $m(a, b) = \# M(a, B)$. Given an ordinal preference profile $\succ$, the assignment corresponding to the profile $\omega = (\omega_i)_{i \in N}$ of agents' eating speeds is defined by the following recursive procedure. Let $A^0 = A$, $y^0 = 0$, $P^0 = [0]$, the $n \times n$ matrix of zeros. Suppose that A^0, y^0, P^0, ..., A^{s-1}, y^{s-1}, P^{s-1} are already defined. For any $a \in A^{s-1}$ define

$$y^s(a) = \min \left\{ y \;\middle|\; \sum_{i \in M(a, A^{s-1})} \int_{y^{s-1}}^{y} \omega_i(t)\, dt + \sum_{i \in N} p_{ia}^{s-1} = 1 \right\}$$

$$(y^s(a) = +\infty, \text{ if } M(a, A^{s-1}) = \varnothing). \tag{8}$$

Define now

$$y^s = \min_{a \in A^{s-1}} y^s(a)$$

$$A^s = A^{s-1} \backslash \{a \mid y(a) = y^s\}$$

$$P^s : p_{ia}^s = \begin{cases} p_{ia}^{s-1} + \int_{y^{s-1}}^{y^s} \omega_i(t)\, dt, & \text{if } \; i \in M(a, A^{s-1}) \\ p_{ia}^{s-1}, & \text{otherwise.} \end{cases}$$

By the construction, $A^s \subsetneq A^{s-1}$ for all s, hence $A^n = 0$ and $P^n = P^{n+1} = \cdots$. The matrix P^n is the random assignment corresponding to the profile of eating speeds $\omega = (\omega_i)_{i \in N}$ and the preference profile $\succ$: $P_\omega(\succ) = P^n$.

THEOREM 1. *Fix a preference profile $\succ$ in $\mathscr{A}^N$. For every profile of eating speed functions $\omega = (\omega_i)_{i \in N}$, the random assignment $P_\omega(\succ)$ is ordinally efficient. Conversely, for every ordinally efficient random assignment P at $\succ$, there exists a profile $\omega = (\omega_i)_{i \in N}$ such that $P = P_\omega(\succ)$.*[8]

6. THE PROBABILISTIC SERIAL ASSIGNMENT

DEFINITION 3. The probabilistic serial assignment at a given preference profile $\succ$ is the random assignment corresponding to the profile of *uniform eating speeds*: $\omega_i(t) = 1$ for all $i \in N$, all t, $0 \leqslant t \leqslant 1$. It is denoted $\mathrm{PS}(\succ)$.

[8] It is easy to see that Theorem 1 is preserved if we restrict attention to profiles of eating speed functions such that at any instant t, only one among $\omega_i(t)$ is strictly positive.

In view of Theorem 1, the PS assignment is the simplest fair selection from the set of ordinally efficient assignments at a given preference profile: the mechanism PS is *anonymous*, that is to say the mapping $\succ \to \text{PS}(\succ)$ is symmetric from the n preferences $\succ_i$ to the n assignments P_i. Similarly the RP assignment (6) is the most natural fair selection from the set of ex post efficient assignments (in view of (5)).

In fact, the PS mechanism is the *only* equitable mechanism we can construct in this fashion. That is to say, whenever we use a simultaneous eating algorithm (the same for all profiles) to construct an anonymous assignment rule, we must end up with the PS mechanism:

LEMMA 4. *Fix at vector of eating speeds $\omega = (\omega_1, ..., \omega_n)$. Let P be the mechanism derived from ω at all profiles. P is anonymous if and only if it coincides with PS.*

We now compare the PS and RP assignment matrices, and their welfare implications in a few examples. Example (1) in Section 2 with four agents was discussed in the previous section: there the PS assignment (3) stochastically dominates the RP assignment (2). In problems with two or three agents, this configuration cannot happen.

In the case of two agents, if their top choices are different the only ex-post efficient assignment gives them these objects. Otherwise, equal treatment of equals implies that each agent gets each good with probability 0.5. Naturally, both the RP and PS assignments do exactly this.

Next we look at three agents problems. Up to relabeling the objects, and/or the agents, there are only two profiles of deterministic preferences at which the RP and PS assignments differ[9], namely:

$$\begin{array}{l} a \succ_1 b \succ_1 c \\ a \succ_2 c \succ_2 b \\ b \succ_3 a,\, c \end{array} \qquad PS = \begin{array}{ccc} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{array} \qquad RP = \begin{array}{ccc} \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{5}{6} & \frac{1}{6} \end{array} \tag{9}$$

Note that neither assignment RP or PS stochastically dominates the other, yet the preferences of the agents between the two assignments are unambiguous:

agent 1 prefers PS over RP as $(\frac{1}{2}a + \frac{1}{4}b + \frac{1}{4}c)\ sd(\succ_1)(\frac{1}{2}a + \frac{1}{6}b + \frac{1}{3}c)$
agent 3 prefers RP over PS as $(\frac{5}{6}b + \frac{1}{6}c)\ sd(\succ_3)(\frac{3}{4}b + \frac{1}{4}c)$
agent 2 receives the same allocation under PS and RP.

[9] The complete listing of the ten profiles of deterministic preferences for three agents problems is given in the proof of Lemma 2, in the Appendix.

It may also happen that the individual random allocations under RP and PS are not comparable by stochastic dominance, and this holds true for every agent. At such a profile of deterministic preferences (an example is given below), there is a compatible profile of VNM utilities such that the RP assignment is ex ante (strictly) Pareto superior to the PS assignment, and there is another compatible profile such that the PS assignment is ex ante (strictly) Pareto superior to the RP one.

The example has six agents with the following preferences:

agents 1, 2: $a \succ b \succ c \succ d \succ e \succ f$

agent 3: $c \succ a \succ b \succ d \succ e \succ f$

agent 4: $c \succ e \succ f \succ d \succ a \succ b$

agents 5, 6: $e \succ f \succ c \succ d \succ a \succ b$.

The assignment PS($\succ$) is as follows:

	a	b	c	d	e	f
1, 2	1/2	1/3	0	1/6	0	0
3	0	1/3	1/2	1/6	0	0
4	0	0	1/2	1/6	0	1/3
5, 6	0	0	0	1/6	1/2	1/3

The assignment RP($\succ$) is as follows:

	a	b	c	d	e	f
1, 2	11/24	5/12	0	1/8	0	0
3	1/12	1/6	1/2	1/4	0	0
4	0	0	1/2	1/4	1/12	1/6
5, 6	0	0	0	1/8	11/24	5/12

Consider agent 3, who gets one of $\{c, a\}$, his or her two top objects, with probability 7/12 under RP versus only 1/2 under PS; on the other hand, agent 3 gets one of $\{c, a, b\}$, his or her top three objects, with probability 5/6 under PS versus only 3/4 under RP. Therefore agent 3's two assignments are not comparable by stochastic dominance. Similar arguments establish the same property for agent 1 and, by symmetry, for all other agents.

With four agents, only a slightly weaker example can be found, where three out of four agents have noncomparable assignments whereas the fourth agent gets the same assignment:

$$\begin{array}{ll} agents\ 1, 2: & a \succ b \succ c \succ d \\ agent\ 3: & c \succ a \succ b \succ d \\ agent\ 4: & c \succ d \succ a \succ b \end{array}$$

$$\begin{matrix} 11/24 & 5/12 & 0 & 1/8 \\ 11/24 & 5/12 & 0 & 1/8 \\ 1/12 & 1/6 & 1/2 & 1/4 \\ 0 & 0 & 1/2 & 1/2 \end{matrix} = \text{RP}; \qquad \begin{matrix} 1/2 & 1/3 & 0 & 1/6 \\ 1/2 & 1/3 & 0 & 1/6 \\ 0 & 1/3 & 1/2 & 1/6 \\ 0 & 0 & 1/2 & 1/2 \end{matrix} = \text{PS}$$

7. FAIRNESS AND INCENTIVES: COMPARING RANDOM PRIORITY AND PROBABILISTIC SERIAL

We now compare the probabilistic serial and random priority assignments (resp. mechanisms) by means of the no envy (resp. strategyproofness) properties.

The profile (9) (Section 6) reveals two interesting facts. First, at the RP assignment agent 1 may envy the allocation of agent 3: for some utility functions u_1 compatible with $\succ_1$ we have

$$u_1 \cdot RP_3 = \tfrac{5}{6}\, u_1(b) + \tfrac{1}{6}\, u_1(c) > \tfrac{1}{2}\, u_1(a) + \tfrac{1}{6}\, u_1(b) + \tfrac{1}{3}\, u_1(c) = u_1 \cdot RP_1 \tag{10}$$

(e.g., take $u_1(a) = 10$, $u_1(b) = 9$, $u_1(c) = 0$). By contrast, no agent can be envious at the PS assignment for any compatible VNM utilities.

The second fact is that agent 3 can profitably misreport his or her preferences at the PS assignment when his or her preferences are $b \succ_3 a \succ_3 c$. If agent 3 reports $a \succ_3^* b \succ_3^* c$ he or she receives

$$PS_3(\succ_1, \succ_2, \succ_3^*) = \tfrac{1}{3}\, a + \tfrac{1}{2}\, b + \tfrac{1}{6}\, c.$$

For some utility functions u_3 compatible with agent 3's true preferences $\succ_3$, we have

$$\tfrac{1}{3}\, u_3(a) + \tfrac{1}{2}\, u_3(b) + \tfrac{1}{6}\, u_3(b) \succ \tfrac{3}{4}\, u_3(b) + \tfrac{1}{4}\, u_3(c) \tag{11}$$

(e.g., take $u_3(b) = 10$, $u_3(a) = 9$, $u_3(c) = 0$). By contrast, no agent can manipulate the RP assignment by misreporting, for any compatible VNM utilities.

Thus the RP assignment may generate envy and the PS mechanism is not strategyproof. However the possibilities of envy at the RP assignment and of manipulation at the PS one occur only for *some* utility functions compatible with the deterministic preferences in question: inequalities (11),

(12) do not hold for all VNM utilities compatible with $\succ_1$ and $\succ_3$ respectively. In other words the allocation RP_3 does not stochastically dominate RP_1 for the preferences $\succ_1$; nor does agent 3 have a manipulation after which his or her allocation PS_3^* stochastically dominates PS_3. This suggests the following two definitions.

DEFINITION 4. We say that a random assignment P, $P \in \mathscr{R}$, is envy-free (resp. weakly envy-free) at a profile $\succ$ in $\mathscr{A}^N$ if we have for all $i, j \in N$:

No envy: $P_i \, sd(\succ_i) \, P_j$

Weak no envy: $P_j \, sd(\succ_i) \, P_i \Rightarrow P_i = P_j$.

DEFINITION 5. Given a mechanism $P(\cdot)$, namely a mapping from $\mathscr{A}^N$ into $\mathscr{R}$, we define:

Strategyproofness: $P_i(\succ) \, sd(\succ_i) \, P_i(\succ |^i \succ_i^*)$ for all $i \in N$, $\succ_i^* \in \mathscr{A}$, $\succ \in \mathscr{A}^N$

Weak strategyproofness: $\{P_i(\succ |^i \succ_i^*) \, sd(\succ_i) \, P_i(\succ) \Rightarrow P_i(\succ |^i \succ_i^*) = P_i(\succ)\}$ for all $i \in N$, $\succ_i^* \in \mathscr{A}$, $\succ \in \mathscr{A}^N$.

Our definitions of no envy and strategyproofness are standard if all agents are endowed with VNM utilities (see, e.g., Roth and Rothblum [15]). The weaker versions are sufficient if our agents compare lotteries over A by their partial ordering of stochastic dominance.

PROPOSITION 1.

(i) *For any profile* $\succ$ *in* $\mathscr{A}^N$, *the assignment* $PS(\succ)$ *is envy-free; the assignment* $RP(\succ)$ *is weakly envy-free but may not be envy-free for* $n \geqslant 3$.

(ii) *The RP mechanism is strategyproof; the PS mechanism is weakly strategyproof but not strategyproof for* $n \geqslant 3$.

8. AN IMPOSSIBILITY RESULT

To characterize our two mechanisms RP and PS with the help of their efficiency, equity and incentives properties is easy in the case of two or three agents. But we run into a severe impossibility result when the number of agents is four or more.

As already noted in Section 6, with two agents PS and RP coincide, and are characterized by the combination of (ex post) efficiency and equal treatment of equals (namely, $\succ_i = \succ_j \Rightarrow P_i = P_j$). Two interesting characterizations are available in the three agents case.

PROPOSITION 2. *Assume $n=3$. Then the random priority mechanism is characterized by the combination of three axioms: ordinal efficiency, strategy-proofness, and equal treatment of equals.*

The probabilistic serial mechanism is characterized by the combination of three axioms: ordinal efficiency, no envy, and weak strategyproofness.

A corollary of Proposition 2 and Lemma 2 is the incompatibility of the three requirements: ex post efficiency, strategyproofness and no envy. Indeed, no envy implies equal treatment of equals, because our mechanisms only take deterministic preferences into account. Therefore a mechanism meeting the three properties listed would have to be both RP and PS.

For problems involving four agents or more, the impossibility result is more severe.

THEOREM 2. *Assume $n \geqslant 4$. Then there is no mechanism—no mapping from $\mathscr{A}^N$ into $\mathscr{R}$—meeting the three following requirements: ordinal efficiency, strategyproofness, and equal treatment of equals.*

This result is interestingly similar to Zhou's theorem (Zhou [19]), although neither result implies the other. Zhou works in the class of mechanisms eliciting a full VNM utility function from every agent. This class is considerably larger than the class of mechanisms considered here. Zhou shows the incompatibility of equal treatment of equals, strategyproofness, and ex ante efficiency.

9. CONCLUDING COMMENTS

We list a handful of variants of our random assignment model; for all but one, the analysis of ordinal efficiency as well as the definitions and comparison of the random priority and probabilistic serial assignments extend almost verbatim.

(a) *Different Number of Objects and Agents*

Suppose that we have m objects, n agents and $m > n$. Then a random assignment P is a nonnegative $n \times m$ matrix whose rows sum to one and whose columns sum to at most one. The definition of the three notions of efficiency (Section 4) and the two characterizations of ordinal efficiency (Section 5) are preserved: in the simultaneous eating algorithm, each agent receives a speed function ω_i of which the integral sums to 1. The definition of the PS assignment (Section 6) and its comparison with the RP assignment (Proposition 1 in Section 7) remain true.

Next consider the case with more agents than objects: $n > m$. Now a random assignment P is a nonnegative $n \times m$ matrix whose rows sum to m/n and whose columns sum to one. The characterization of ordinal efficiency in terms of the acyclicity of the relation τ (Lemma 3) remains true, that in terms of the simultaneous eating algorithms is preserved as well, provided we allow a total eating capacity m/n per agent ($\int \omega_i = m/n$).

Alternatively, the assignment problem with $m < n$ can be transformed into an assignment problem with n objects by adding $(n-m)$ copies of a null object. The difference with our initial model is that some objects are "objectively" identical: as discussed in our next comment, this type of indifferences poses no special problem.

(b) *Objective Indifferences*

Suppose that some objects are objectively identical in the sense that all agents are indifferent between them. Formally we assume that for all a, b, the statement "agent i is indifferent between a and b" holds for all i or for no i.

All our definitions and results extend almost verbatim to this case. For instance consider the two characterizations of ordinal efficiency in Section 5. The relation $\tau(P, \succ)$ (7) is defined in the same way, and $a\tau b$ never holds between two identical objects. As for the simultaneous eating algorithm, it is defined up to the (inconsequential) choice among identical objects, so that Theorem 1 is preserved, and the probabilistic serial assignment is unambiguously defined. The random priority assignment is similarly well defined.

The careful reader will check that Proposition 1 still holds true.

(c) *Subjective Indifferences*

Suppose individual preferences vary in the classical domain of (complete and transitive) preferences: that is, agent i may be indifferent between objects a, b whereas other agents are not. An ordinal mechanism would elicit the full preference relation and could in particular use subjective indifferences to improve efficiency.

It is not clear, however, how this should be done. Think of the random priority mechanism; an agent whose turn it is to choose may be indifferent between several objects: the mechanism must define the tie breaking rule (using, presumably, the information about the full profile of ordinal preferences) and do so as efficiently as possible.

The definition of the simultaneous eating algorithms raises similar difficulties. And the link of such algorithms to ordinal efficiency is wholly unclear. We submit that the random assignment problem with subjective indifferences is as interesting as it is challenging and worthy of further research.

(d) *Opting Out*

In some examples of the assignment problem, an agent can always opt out (claim the null object), namely refuse to accept certain objects (less desirable than the null object). Bogomolnaia and Moulin [3] study a simple assignment problem with opting out: all n agents have the same ordinal ranking of the n objects, but differ in their ranking of the null object with respect to the real objects. An example is the scheduling of jobs by a single server: every agent prefers to be served earlier than later, but agents have different "deadlines."

In that model, a random assignment is a substochastic matrix (the sum of any row or any column is at most one) and the set of ordinally efficient assignments is easy to describe (see Lemma 3.3 in Bogomolnaia and Moulin [3]). Interestingly, the PS assignment (also defined by the equal eating speeds algorithm) is equal to or stochastically dominates the RP assignment. Moreover, the PS mechanism is strategyproof. Finally the PS mechanism is characterized by the combination of ordinal efficiency, strategyproofness, and equal treatment of equals and the PS assignment is characterized by ordinal efficiency plus no envy.

Back to our model where different agents may rank the real objects differently, it is straightforward to extend the definition of ordinal efficiency, its two characterizations (Section 5), and the definition of the PS (and RP) assignment when opting out is possible. Proposition 1 is preserved as well.

APPENDIX: PROOFS

1. *Proof of Lemma* 2

Statement i. Suppose P is stochastically dominated by Q at $\succ$ (Definition 2). As noted immediately before Definition 2 this implies $u_i \cdot P_i \geqslant u_i \cdot Q_i$ for all i; moreover, $P_i \neq Q_i$ implies that the corresponding inequality is strict so that P is ex ante Pareto inferior to Q.

We give an example with $n=3$ of an ordinally efficient random allocation P that is not ex ante efficient. Consider the utility profile in footnote 4. It is compatible with the unanimous ordinal preferences $a \succ_i b \succ_i c$. The random priority assignment P is given by $p_{ix} = \frac{1}{3}$ for $i = 1, 2, 3$ and $x = a, b, c$. It is not ex ante efficient for the profile u given in footnote 4, yet every random assignment in $\mathscr{R}$ is ordinally efficient because the three relations $sd(\succ_i)$ coincide.

Statement ii. Suppose P is not ex post efficient at $\succ$. Consider a decomposition of P as a convex combination of deterministic assignments:

$$P = \sum_{\Pi \in \mathscr{D}} \lambda_\Pi \cdot \Pi.$$

By Lemma 1 and statement ii in Definition 1, there is an element Π in $\mathscr{D}$ that is Pareto inferior at $\succ$ and such that $\lambda_{\Pi} > 0$. Let Π' be a deterministic assignment Pareto superior to Π. Upon replacing Π with Π' in the summation, we obtain a random assignment that stochastically dominates P (note that stochastic dominance in $\mathscr{R}$ is preserved by convex combinations).

The four agents example in Section 2 shows an ex post efficient assignment that is not ordinally efficient. The preference profile $\succ$ is given by (1) and the random assignment (2) equals $RP(\succ)$. By Definition 1, this random assignment is ex post efficient. However it is stochastically dominated by the feasible random assignment given by (3).

Finally, we must show that for $n = 3$ every ex post efficient assignment is ordinally efficient as well. To this end we note first that, up to relabeling the objects and/or the agents, there are exactly ten different profiles of deterministic preferences:

$$
\begin{array}{llll}
 & a \succ_1 b, c & & a \succ_1 b \succ_1 c \\
\text{type 1 (2 profiles)} & b \succ_2 a, c & \text{type 2} & a \succ_2 b \succ_2 c \\
 & c \succ_3 a, b & & a \succ_3 b \succ_3 c \\[1em]
 & a \succ_1 b \succ_1 c & & a \succ_1 c \succ_1 b \\
\text{type 3} & a \succ_2 b \succ_2 c & \text{type 4 (2 profiles)} & a \succ_2 c \succ_2 b \\
 & a \succ_3 c \succ_3 b & & b \succ_3 a, c \\[1em]
 & a \succ_1 b \succ_1 c & & a \succ_1 b \succ_1 c \\
\text{type 5 (2 profiles)} & a \succ_2 b \succ_2 c & \text{type 6 (2 profiles)} & a \succ_2 c \succ_2 b \\
 & b \succ_3 a, c & & b \succ_3 a, c
\end{array} \tag{12}
$$

In type 1 the only ex post efficient assignment is PS = RP. In type 2 any feasible assignment is ordinally efficient. In type 3 every (deterministic) priority assignment $Prio(\sigma, \succ)$ has $p_{3b} = 0$; hence every ex post efficient assignment has $p_{3b} = 0$. The latter implies ordinal efficiency: if Q stochastically dominates P, we must have $p_{ia} \leqslant q_{ia}$ for all i. Hence all three inequalities are equalities; next $p_{ia} + p_{ib} \leqslant q_{ia} + q_{ib}$ for $i = 1, 2$ implies $p_{ib} = q_{ib}$ for $i = 1, 2$ and hence $Q = P$. Next, consider type 4, where every priority assignment, and hence every ex post-efficient assignment as well, has $p_{3b} = 1$. This, in turn, implies ordinal efficiency.

Consider type 5, where every priority assignment, and hence every ex post efficient assignment as well, has $p_{3a} = 0$, which implies ordinal efficiency (by an argument similar to that for type 3 above). Finally in type 6, every priority assignment, and hence every ex post efficient assignment as well, has $p_{2b}=p_{3a}=0$, implying ordinal efficiency by the same kind of argument again: if Q stochastically dominates P we have successively

$$p_{ia} \leqslant q_{ia} \quad \text{for } i=1,2 \Rightarrow p_{ia}=q_{ia} \quad \text{for } i=1,2$$

$$\{p_{1a}+p_{1b} \leqslant q_{1a}+q_{1b}; p_{3b} \leqslant q_{3b}\} \Rightarrow p_{1b}=q_{1b} \quad \text{and} \quad p_{3b}=q_{3b}$$

and $Q=P$ as desired.

2. *Proof of Lemma* 3

Statement only if. Suppose the relation $\tau(P, \succ)$, denoted τ for simplicity, has a cycle:

$$a_2 \tau a_1; a_3 \tau a_2; ...; a_K \tau a_{K-1}; a_K=a_1$$

(we assume, without loss of generality, that the objects a_k, $k=1, ..., K-1$, are all different). By definition of τ, we can construct a sequence $i_1, ..., i_{K-1}$ in N:

$$p_{i_1a_1}>0 \quad \text{and} \quad a_2 \succ_{i_1} a_1;$$

$$p_{i_2a_2}>0 \quad \text{and} \quad a_3 \succ_{i_2} a_2; ...; p_{i_{K-1}a_{K-1}}>0 \quad \text{and} \quad a_K \succ_{i_{K-1}} a_{K-1}$$

(note that the agents i_k, $k=1, ..., K-1$, may not be all different). Choose δ such that:

$$\delta>0 \quad \text{and} \quad \delta \leqslant p_{i_ka_k} \qquad \text{for} \quad k=1, ..., K-1.$$

Then define a matrix Q as follows:

$$Q=P+\Delta \qquad \text{where} \quad \delta_{i_ka_k}=-\delta; \quad \delta_{i_ka_{k+1}}=+\delta \quad \text{for } k=1, ..., K-1$$
$$\text{and} \quad \delta_{ia}=0 \text{ otherwise.}$$

By construction, Q is a bistochastic matrix, $Q \in \mathscr{R}$; moreover, Q stochastically dominates P, because one goes from P_{i_k} to Q_{i_k} by shifting some probability from object a_k to the preferred object a_{k+1} (and if the same agent appears more than once, we use the transitivity of stochastic dominance).

Statement if. Suppose P in $\mathscr{R}$ is stochastically dominated at $\succ$ by Q in $\mathscr{R}$. Let i_1 be an agent such that $Q_{i_1} \neq P_{i_1}$ (Definition 2). By definition of the relation $sd(\succ_{i_1})$, there exist two objects a_1, a_2 such that

$$a_2 \succ_{i_1} a_1 \qquad q_{i_1a_1}<p_{i_1a_1} \qquad \text{and} \qquad p_{i_1a_2}<q_{i_1a_2}.$$

In particular, $a_2 \tau(P, \succ)\, a_1$. Next by feasibility of Q, there exists an agent i_2 such that $q_{i_2 a_2} < p_{i_2 a_2}$. Repeating the argument, we find a_3 such that

$$a_3 \succ_{i_2} a_2 \qquad \text{and} \qquad p_{i_3 a_3} < q_{i_3 a_3},$$

and hence $a_3 \tau(P, \succ)\, a_2$, and so on, until by finiteness of A and N we find a cycle of the relation τ.

3. *Proof of Theorem* 1

Fix an ordinal preference profile $\succ$. The set $\{P_\omega(\succ) \mid \omega \in W^n\}$ coincides with the set of all random assignments ordinally efficient with respect to $\succ$.

(i) Any $P_\omega(\succ)$ is ordinally efficient. We prove it by contradiction. Suppose that for some ω $P_\omega(\succ)$ is not ordinally efficient. By Lemma 3 we can find a cycle in the relation τ:

$$a^0 \tau\, a^1, ..., a^{r-1} \tau\, a^r, ..., a^R \tau\, a^0.$$

Let i_r be an agent such that $a^{r-1} \succ_{i_r} a^r$ and $p_{i_r a^r} > 0$ ($r \in 1, ..., R+1$, with the convention $a^{R+1} = a^0$). Let s^r be the first step s in our simultaneous eating algorithm when the agent i_r starts to acquire good a^r, i.e., the least s for which $p^s_{i_r a^r} \neq 0$.

Since in the algorithm p_{ia} can change from P^{s-1} to P^s only if $i \in M(a, A^{s-1})$, we deduce that at the step s^r good a^{r-1} has to be already fully distributed, i.e., $a^{r-1} \notin A^{s^r - 1}$. Thus, $s^{r-1} < s^r$ for all $r = 1, ..., R+1$, which is a contradiction since $a^0 = a^{R+1}$.

(ii) Any ordinally efficient assignment P can be constructed using a simultaneous eating algorithm for some vector ω of eating speeds. Let $\bar{A}^0 = A$, $B^1 =$ the set of maximal elements of $\bar{A}^0$ under τ, i.e., $B^1 = \{a \in \bar{A}^0 \mid \nexists b \in \bar{A}^0 : b \tau a\}$. Let $B^s = \{a \in \bar{A}^{s-1} \mid \nexists b \in \bar{A}^{s-1} : b \tau a\}$, $\bar{A}^s = \bar{A}^{s-1} \setminus B^s$, ... This sequence stops at a step S, for which $B^S = \bar{A}^{S-1}$ (note that $\bar{A}^n = \varnothing$ so $\varnothing = B^{n+1} = \bar{A}^{n+1} = \cdots$).

Define now for all $s = 1, ..., S$:

$$\text{for} \quad \frac{s-1}{S} \leqslant t \leqslant \frac{s}{S} \qquad \omega_i(t) \overset{\text{def}}{=} \begin{cases} S p_{ia}, & \text{if} \quad a \in B^s \quad \text{and} \quad i \in M(a, A^{s-1}) \\ 0, & \text{otherwise.} \end{cases}$$

We will check that P is the result of the simultaneous eating algorithm with eating speeds $(\omega_1, ..., \omega_n)$ and that $\bar{A}^0, ..., \bar{A}^s$ coincide with $A^0, ..., A^s$ from this algorithm. Indeed, let $a \in B^s$. By the maximality of a in $\bar{A}^{s-1}$, $p_{ia} > 0$

implies $i \in M(a, \bar{A}^{s-1})$. Thus, $M(a, \bar{A}^{s-1}) \neq \varnothing$ and $p_{ia} = 0$ whenever $i \notin M(a, \bar{A}^{s-1})$. For $s = 1$ we obtain:

$$y^1(a) = \begin{cases} \dfrac{1}{S}, & \text{if } \ a \in B^1 \\ \infty, & \text{if } \ a \notin B^1. \end{cases}$$

Hence, $y^1 = 1/S$, $\bar{A}^1 = A^1$, and P^1 is such that

$$p^1_{ia} = \begin{cases} p_{ia}, & \text{if } \ a \in B^1 \\ 0, & \text{if } \ a \notin B^1. \end{cases}$$

We proceed by induction. Suppose that

$$p^{s-1}_{ia} = \begin{cases} p_{ia}, & \text{if } \ a \in B^1 \cup \cdots \cup B^{s-1} \quad \text{and} \quad y^{s-1} = \dfrac{s-1}{S} \\ 0, & \text{otherwise.} \end{cases}$$

We have for any a in $\bar{A}^{s-1}$:

$$\sum_{i \in M(a, \bar{A}^{s-1})} \int_{s-1/S}^{y} \omega_i(t)\, dt + \underbrace{\sum_{i \in N} p^{s-1}_{ia}}_{=0}$$

$$= \begin{cases} \displaystyle\sum_{i \in M(a, \bar{A}^{s-1})} \int_{s-1/S}^{y} S p_{ia}\, dt = [Sy - (s-1)] \\ \qquad \cdot \underbrace{\sum_{M(a, \bar{A}^{s-1})} p_{ia}}_{=1} = Sy - (s-1), & \text{if } \ a \in B^s \\ 0, \quad \text{if } \ a \notin B^s. \end{cases}$$

So,

$$y^s(a) = \begin{cases} \dfrac{s}{S}, & \text{if } \ a \in B^s \\ \infty, & \text{otherwise.} \end{cases}$$

Thus $y^s = s/S$, $\bar{A}^s = A^s$ and

$$p^s_{ia} = \begin{cases} p_{ia}, & \text{if } \ a \in B^1 \cup - \cup B^s \\ 0, & \text{otherwise.} \end{cases}$$

4. *Proof of Lemma* 4

We fix ω and P as in the statement of the lemma and assume that P is anonymous. We fix a preference profile $\succ$.

The partial assignment obtained under PS at any moment $t \in [0, 1]$ is anonymous, so under $\succ$ or any of its (agents) permutations, objects a_1, a_2, ..., a_k, ..., a_n are eaten away in the same order and at the same instants $0 < x_1 \leqslant x_2 \leqslant \cdots \leqslant x_k \leqslant \cdots \leqslant x_n = 1$. Note also that under PS, an agent can change the good he or she eats only at one of the instants s_k and that the set of agents who eat a given good can only expand with time.

Define $S(a_k)$ to be the set of agents who eat good a_k in $[x_{k-1}, x_k]$. If $|S(a_k)| = 1$ then a_k is entirely assigned to one agent and $x_k = 1 = x_n$. Thus, $|S(a_k)| \geqslant 2$ whenever $x_k < x_n$.

Step 1. Suppose there exist instants $0 < y_1 \leqslant y_2 \leqslant \cdots \leqslant y_k \leqslant \cdots \leqslant y_n = 1$ such that at y_k all agents get under P exactly the x_k fraction of their unit share of goods, i.e., $\int_0^{y_k} \omega_i(\tau)\, d\tau = x_k\ \forall i, k$. Then P coincides with PS.

Indeed, suppose that assignments are the same at x_1, ..., x_{k-1} under PS and at y_1, ..., y_{k-1} under P (where $x_0 = y_0 = 0$). Under PS during $[x_{k-1}, x_k]$ each agent eats his or her best among the goods still available a_k, ..., a_n, and the fraction $x_k - x_{k-1}$ eaten by everyone will not exhaust any good before x_k. Since $x_k - x_{k-1}$ is exactly the fraction each agent eats during the interval $[y_{k-1} - y_k]$ under P, they will end up at y_k with the same partial assignment as at x_k under PS.

Step 2. Check that such y_1, ..., y_n exist. Define

$$\bar{t}_i(k) = \max\left\{t\colon \int_0^t \omega_i(\tau)\, d\tau \geqslant x_k\right\} \qquad \underline{t}_i(k) = \min\left\{t\colon \int_0^t \omega_i(\tau)\, d\tau \geqslant x_k\right\}$$

$$\bar{t}(k) = \min_i \bar{t}_i(k), \qquad \underline{t}(k) = \max_i \underline{t}_i(k),$$

i.e., $[\underline{t}_i(k), \bar{t}_i(k)]$ is the largest interval during which the total fraction of goods eaten by an agent i stays equal to x_k.

Proceed by induction on k. Suppose that under P all agents are able to eat exactly the fractions x_1, ..., x_{k-1} by the dates y_1, ..., y_{k-1} respectively (where $x_0 = y_0 = 0$). If $\underline{t}(k) \leqslant \bar{t}(k)$ then choose any $y_k \in [\underline{t}(k), \bar{t}(k)]$. Suppose that $\underline{t}(k) > \bar{t}(k)$.

Consider the permutations $\succ^1$ and $\succ^2$ of $\succ$, such that agents 1 and 2 are in $S(a_k)$, $\bar{t}(k) = \bar{t}_1(k)$ and $\underline{t}(k) = \underline{t}_2(k)$ under $\succ^1$, and $\succ^2$ is obtained from $\succ^1$ by exchanging agents 1 and 2. We have

$$\sum_{i \in S(a_k)} \int_{y_{k-1}}^{\bar{t}(k)} \omega_i(\tau)\, d\tau < |S(a_k)|\ (x_k - x_{k-1}) = \text{amount of } a_k \text{ left}$$

$$< \sum_{i \in S(a_k)} \int_{y_{k-1}}^{\underline{t}(k)} \omega_i(\tau)\, d\tau$$

and for any good a_j, $j > k$,

$$\sum_{i \in S(a_j)} \int_{y_{k-1}}^{\bar{t}(k)} \omega_i(\tau)\, d\tau \leqslant |S(a_j)|\, (x_k - x_{k-1}) \leqslant \text{amount of } a_j \text{ left.}$$

Moreover, the equality is possible only if $x_j = x_k$.

Thus, under $\succ^1$ and $\succ^2$ no good among $a_k, ..., a_n$ is eaten away before $\bar{t}(k)$ and good a_k will be exhausted at some dates $s^1, s^2 \in (\bar{t}(k), \underline{t}(k))$. But for any s in this interval, under $\succ^1 (\succ^2)$ the fraction of goods agent 1 (2) gets by time s is larger than x_k, while the fraction of goods agent 2 (1) gets by the time s is smaller than x_k.

By our induction hypothesis, all agents get exactly the same partial assignment at x_{k-1} under PS and at y_{k-1} under P. As a result, agent 1 will get more and agent 2 less than x_k of good a_1 under $\succ^1$, while agent 2 will get more and agent 1 less than x_k of good a_1 under $\succ^2$. This contradicts the anonymity of P.

5. *Proof of Proposition* 1

Step 1. *The PS assignment is envy-free.* Fix $\succ$ in $\mathscr{A}^N$, i in N and label A in such a way that $a \succ_i b \succ_i c \succ \cdots$. Consider the algorithm in Section 5 keeping in mind $\omega_i(t) = 1$ for all i, t. Let s_1 be the step at which a is fully allocated, namely

$$a \in A^{s_1 - 1} \backslash A^{s_1}.$$

Because $i \in M(a, A^s)$ as long as $s \leqslant s_1 - 1$, we have

$$p_{ia}^{s_1} = y^{s_1} \geqslant p_{ja}^{s_1} \qquad \text{for all} \quad j \in N.$$

Because a is fully allocated at s_1, these numbers are respectively the ia and ja entries of $PS(\succ) = P$, so that $p_{ia} \geqslant p_{ja}$. Next we let s_2 be the step at which $\{a, b\}$ is fully allocated

$$\{a, b\} \cap A^{s_2 - 1} \neq \varnothing \qquad \{a, b\} \cap A^{s_2} = \varnothing.$$

Note that $s_1 \leqslant s_2$ and that $i \in M(a, A^s) \cup M(b, A^s)$ for $s \leqslant s_2 - 1$. Hence

$$p_{ia} + p_{ib} = p_{ia}^{s_2} + p_{ib}^{s_2} = y^{s_2} \geqslant p_{ja}^{s_2} + p_{jb}^{s_2} = p_{ja} + p_{jb} \qquad \text{for all} \quad j \in N.$$

Repeating this argument we find that P_i stochastically dominates P_j at $\succ_i$, as desired.

Step 2. *The PS mechanism is weakly strategyproof.* In the simultaneous eating algorithm with $\omega_i(t) = 1$ for all i, all t, $0 \leqslant t \leqslant 1$, we introduce the following notations: $N(a, t)$ is the (possibly empty) set of agents who eat object a at time t: if t is such that for some $s = 1, ..., n : y^{s-1} \leqslant t < y^s$ then

$$N(a, t) = M(a, A^{s-1}) \qquad \text{if} \quad a \in A^{s-1}$$
$$= \varnothing \qquad \text{if} \quad a \notin A^{s-1}.$$

We write $n(a, t)$, the cardinality of $N(a, t)$, and set $t(a)$ to be the time at which a dies, namely:

$$t(a) = \sup\{t \mid n(a, t) \geqslant 1\}.$$

Observe that $n(a, t)$ is nondecreasing in t on $[0, t(a)[$, because once agent i joins $N(a, t)$, he or she keeps eating object a until its exhaustion. Moreover,

$$\int_0^{t(a)} n(a, t)\, dt = 1 \tag{13}$$

because one unit of object a is allocated during the entire algorithm.

We turn to the proof of the claim. Fix $\succ$ in $\mathscr{A}^N$, an agent denoted as agent 1, and a misreport $\succ_1^*$ by this agent. We write $P = PS(\succ)$, $P^* = PS(\succ |^1 \succ_1^*)$, and similarly $N(a, t)$, $N^*(a, t)$, and so on. Finally we label A so that $a \succ_1 b \succ_1 c \succ \cdots$.

We assume $P_1^*\, sd(\succ_1)\, P_1$ and show $P_1^* = P_1$. If $p_{1a} = 1$, this implication is obvious so we assume $p_{1a} < 1$ from now on. Note that, at profile $\succ$, agent 1 is eating a during the whole interval $[0, t(a)[$; hence $p_{1a} = t(a)$. At $\succ^*$, on the other hand, agent 1 is eating a on a subset of $[0, t^*(a)[$. Therefore the assumption $p_{1a} \leqslant p_{1a}^*$ implies $t(a) \leqslant t^*(a)$.

We claim that for all t in $[0, t(a)[$ and all agents i, $i \neq 1$, we have:

$$i \in N(a, t) \Rightarrow i \in N^*(a, t). \tag{14}$$

Suppose there is an agent i, $i \neq 1$, and a time t, $0 \leqslant t < t(a)$ such that

$$i \in N(a, t) \quad \text{and} \quad i \in N^*(x, t), \qquad \text{for some object } x, \quad x \neq a.$$

As object a is available at time t under profile $\succ^*$ (because $t < t^*(a)$), agent i prefers x to a. Hence x is not available at t under $\succ$ (recall that agent i's preferences do not change in the two profiles). This implies $t(x) \leqslant t < t^*(x)$.

Now let B be the set of objects x such that $x \neq a$ and $t(x) < t^*(x)$ (we have just shown that B is nonempty). We pick y in B for which $t(x)$ is minimal. Note that $t(y) < t(a)$ because in the above construction of x we had $t(x) < t(a)$. From $t(y) < t^*(y)$ we deduce that at some date t, $t < t(y)$, there is an agent j such that

$$j \in N(y, t) \qquad \text{and} \qquad j \notin N^*(y, t). \tag{15}$$

Otherwise the inclusion $N(y, t) \subseteq N^*(y, t)$ for t in $[0, t(y)[$ combined with

$$\int_0^{t(y)} n(y, t)\, dt = 1 = \int_0^{t^*(y)} n^*(y, t)\, dt \tag{16}$$

and the fact that $n^*(y, t)$ is nondecreasing in t would contradict our assumption $t(y) < t^*(y)$. Note that agent j in property (15) cannot be agent 1 because $t < t(y) < t(a)$ and agent 1 eats a over the whole internal $[0, t(a)[$ under $\succ$. Let z be the good that agent j eats at date t under $\succ^*: j \in N^*(z, t)$. As object y is available at t under $\succ^*$ (because $t < t(y) < t^*(y)$), agent j prefers z to y. As agent j eats y at t under $\succ$, and his or her preferences are the same in both profiles ($j \neq 1$) object z is no longer available at t under $\succ$. We have shown successively:

$$t < t(y), \qquad t < t^*(z), \qquad \text{and} \qquad t(z) < t.$$

As z is not object a (because $t(z) < t(a)$) this implies $z \in B$ and $t(z) < t(y)$, a contradiction of the definition of y. This establishes (14).

Thus we have shown $N(a, t) \subseteq N^*(a, t)$ for t in $[0, t(a)[$. By an argument used above (see (16)), this implies $t(a) = t^*(a)$ as well as $N(a, t) = N^*(a, t)$ in this interval. Therefore $p^*_{1a} = p_{1a}$ and the eating algorithms under $\succ$ and $\succ^*$ coincide on the interval $[0, t(a)[$.

It should be clear that the above argument can now be repeated: the assumption $P^*_1\ sd(\succ_1)\ P_1$ gives $p^*_{1b} \geqslant p_{1b}$ and we show successively $t(b) \leqslant t^*(b)$, then $N(b, t) \subseteq N^*(b, t)$ on the interval $[0, t(b)[$, implying $t(b) = t^*(b)$ and so on. We leave the details to the reader.

Step 3. *The RP mechanism is strategyproof.* For any ordering σ of N, the priority mechanism $\succ \to Prio(\sigma, \succ)$ is obviously strategyproof. This property is preserved by convex combinations (with fixed coefficients, independent of $\succ$); hence the claim.

Step 4. *The RP assignment is weakly envy-free.* Let $\succ$ be a profile at which $P_2\ sd(\succ_1)\ P_1$, we must show $P_2 = P_1$. We label the outcomes $a_1, a_2, ..., a_n$ so that $a_1 \succ_1 a_2 \succ_1 a_3 \succ_1 \cdots$.

For any ordering σ of N where 1 precedes 2, let $\bar{\sigma}$ be the ordering obtained from σ by permuting 1 and 2. Clearly the pairs $\{\sigma, \bar{\sigma}\}$ form a partition of θ. As $\succ$ is fixed throughout we omit it in the expression $Prio(\sigma, \succ)$.

If 2 gets a_1 in $Prio(\bar{\sigma})$, so does 1 in $Prio(\sigma)$. In $Prio(\sigma)$, 2 cannot get a_1 (for 1 would snatch it before 2 anyway). Therefore in the random assignment $Q = \{Prio(\sigma) + Prio(\bar{\sigma})\}/2$ we have $q_{2a_1} \leqslant q_{1a_1}$. But $RP(\succ)$ is a convex combination of such assignments; therefore $p_{2a_1} \leqslant p_{1a_1}$. From our assumption $P_2\, sd(\succ_1)\, P_1$ we get $q_{2a_1} = q_{1a_1}$ for all pairs $\sigma, \bar{\sigma}$, so that for any such pair

$$\begin{aligned}&\text{either 1 gets } a_1 \text{ in } Prio(\sigma) \text{ and 2 gets } a_1 \text{ in } Prio(\bar{\sigma}), \text{ or}\\ &\text{none of 1, 2 gets } a_1 \text{ in any of } Prio(\sigma) \text{ or } Prio(\bar{\sigma}).\end{aligned} \tag{17}$$

Next consider the allocation of a_2 in $Prio(\sigma)$, $Prio(\bar{\sigma})$. If 2 gets a_2 in $Prio(\bar{\sigma})$, by (17) 1 cannot get a_1 in $Prio(\sigma)$; hence 1 gets a_2 in $Prio(\sigma)$. If 2 gets a_2 in $Prio(\sigma)$, then 1 gets a_1 in $Prio(\sigma)$ (as 1 precedes 2, 1 must get something he or she prefers to a_2), so 2 gets a_1 in $Prio(\bar{\sigma})$ (by (17)) so 1 gets a_2 in $Prio(\bar{\sigma})$.

We conclude that $q_{2a_2} \leqslant q_{1a_1}$ in Q. By assumption $p_{2a_1} + p_{2a_2} \geqslant p_{1a_1} + p_{1a_2}$ and by the above argument, $p_{2a_1} = p_{1a_1}$; hence $p_{2a_2} = p_{1a_2}$ and $q_{2a_2} = q_{1a_1}$ for all pairs $\{\sigma, \bar{\sigma}\}$. For any such pair the allocation of a_1, a_2, $A \backslash \{a_1, a_2\}$ is "symmetric" between σ and $\bar{\sigma}$; e.g., if $Prio(\sigma)$ has $1 \to x$, $2 \to y$ where x, y are a_1, a_2, or $A \backslash \{a_1, a_2\}$, then $Prio(\bar{\sigma})$ has $2 \to x$, $1 \to y$.

We proceed by induction. Let $p_{1a_i} = p_{2a_i}$, $i = 1, ..., k-1$. Suppose also that for any $x, y \in \{a_1, a_2, ..., a_{k-1}, A \backslash \{a_1, ..., a_{k-1}\}\}$ whenever 1 receives x and 2 receives y at $Prio(\sigma)$, 1 receives y and 2 receives x at $Prio(\bar{\sigma})$.

If 2 gets a_k at $Prio(\bar{\sigma})$ then by the induction hypothesis 1 gets an object from $A \backslash \{a_1, ..., a_{k-1}\}$ at $Prio(\sigma)$. Since a_k is the best for him or her in this set and it is available, 1 gets a_k at $Prio(\sigma)$. If 2 gets a_k at $Prio(\sigma)$, then 1 gets a_e, $e < k$, at $Prio(\sigma)$. Then, by the induction hypothesis, 2 gets a_e at $Prio(\bar{\sigma})$. Hence a_k is available for 1 at $Prio(\bar{\sigma})$. But by induction hypothesis, 1 has to get something from $A \backslash \{a_1, ..., a_{k-1}\}$ at $Prio(\bar{\sigma})$, so he or she gets a_k.

It follows that $q_{2a_k} \leqslant q_{1a_k}$. Since $\sum_{i=1}^{k} p_{2a_i} \geqslant \sum_{i=1}^{k} p_{1a_i}$ by assumption and $p_{1a_i} = p_{2a_i}$ by the induction hypothesis, we deduce as above that $p_{1a_k} = p_{2a_k}$ and that the induction hypothesis holds true when $k-1$ is changed to k.

6. *Proof of Proposition* 2

Step 1. *A consequence of ordinal efficiency.* For all $\succ$, all $a, b \in A$, and all $i \in N$

$$\{a \succ_j b \text{ for all } j \in N \backslash i \text{ and } b \succ_i a\} \Rightarrow \{p_{ia} = 0\}. \tag{18}$$

Note that this property holds for any n. Its proof is simple: if $p_{ia} > 0$ we have $b\tau(P, \succ)a$, whence $p_{jb} > 0$ for some j in $N \backslash i$ would imply $a\tau(P, \succ)b$

and make the relation τ cyclic. By Lemma 3 this is impossible, therefore, $p_{jb}=0$ for all $j\neq i$ implying $p_{ib}=1$, contradiction.

Step 2. *Characterization of the RP mechanism.* We consider in turn the ten different profiles listed in the proof of Lemma 2.

For type 1, ordinal efficiency is enough to pick RP. For type 2, ETE is enough.

For type 3, ordinal efficiency yields $p_{3b}=0$ (by (18)). Next ETE implies $p_{1b}=p_{2b}=\frac{1}{2}$. Finally consider the misreport by agent 3, $a \succ_3^* b \succ_3^* c$, by which this agent gets $(\frac{1}{3})a+(\frac{1}{3})b+(\frac{1}{3})c$. Strategyproofness implies $p_{1a}\geqslant\frac{1}{3}$. Conversely, apply this property at a type 2 profile when agent 3 misreports $a\succ c\succ b$ and deduce $p_{1a}\leqslant\frac{1}{3}$. Thus $p_{1a}=\frac{1}{3}=p_{1b}$ and we are done.

For type 4, ordinal efficiency gives $p_{3b}=1$ (by (18)). Next ETE gives $p_{1a}=p_{2a}$, $p_{1c}=p_{2c}$ and we are done.

For type 5, we distinguish two cases. Start with the case $b\succ_3 a\succ_3 c$. By ordinal efficiency and (18) we get $p_{3a}=0$. If 3 misreports $a\succ_3^* b\succ_3^* c$ he or she gets $(\frac{1}{3})a+(\frac{1}{3})b+(\frac{1}{3})c$; hence by strategyproofness we have $p_{3b}\geqslant\frac{2}{3}$. By considering the symmetric misreport by 3 at the type 2 profile, we get $p_{3b}\leqslant\frac{2}{3}$, so $p_{3b}=\frac{2}{3}$. Now RP can be computed entirely by ETE.

For type 5 with $b\succ_3 c\succ_3 a$, we have $p_{3a}=0$ again by ordinal efficiency and (18). Looking at the misreport by 3 from $b\succ c\succ a$ to $b\succ a\succ c$ and vice versa, we get $p_{3b}=\frac{2}{3}$ and are done.

For type 6, distinguish two cases. First assume $b\succ_3 a\succ_3 c$. Ordinal efficiency and (18) imply $p_{3a}=p_{2b}=0$. Next consider agent 1's misreport: $a\succ_1^* c\succ_1^* b$, which makes the reported profile of type 4. Thus $p_{1a}\geqslant\frac{1}{2}$. The symmetrical misreport by agent 1 starting from the type 4 profile gives $p_{1a}=\frac{1}{2}$. Finally consider agent 3's misreport $a\succ_3 b\succ_3 c$, making the reported profile of type 3, and its symmetric misreport: we obtain $p_{3b}=\frac{5}{6}$ and the matrix is now entirely determined.

In a type 6 preference with $b\succ_3 c\succ_3 a$, we deduce as above $p_{3a}=p_{2b}=0$ and $p_{1a}=\frac{1}{2}$. Finally the misreport $b\succ_3^* a\succ_3^* c$ and its symmetric misreport yield $p_{3b}=\frac{5}{6}$.

Step 3. *Characterization of PS.* First we note that no envy implies equal treatment of equals. Suppose at some profile $\succ$ we have $\succ_1=\succ_2$ but $P_1\neq P_2$. Then for some utility function u compatible with this common ordering of A, we have $u\cdot P_1\neq u\cdot P_2$. Hence one of the agents 1, 2 envies the other if they both have utility u.

Next we look in turn at the ten types of profiles from the proof of Lemma 2. For types 1, 2, and 4, the argument of Step 2 is repeated.

For type 3, ordinal efficiency gives $p_{3b}=0$ and no envy gives $p_{ia}=\frac{1}{3}$ for $i=1, 2, 3$. Then equal treatment of equals gives $p_{1b}=p_{2b}=\frac{1}{2}$.

For type 5 and $b \succ_3 a \succ_3 c$, we have $p_{3a} = 0$ by ordinal efficiency and $p_{1a} = p_{2a} = \frac{1}{2}$ by equal treatment of equals. Next we apply no envy between agents 1 and 3:

$$\{p_{3b} + p_{3a} \geqslant p_{1b} + p_{1a} \text{ and } p_{1a} + p_{1b} \geqslant p_{3a} + p_{3b}\} \Rightarrow p_{3b} = p_{1b} + \tfrac{1}{2}.$$

Because $p_{1b} = p_{2b}$, this gives $p_{1b} = \frac{1}{6}$ and we are done.

For type 5 and $b \succ_3 c \succ_3 a$ we have $p_{3a} = 0$ and $p_{1a} = p_{2a} = \frac{1}{2}$ as above. But no envy alone is not enough to pin down the PS assignment. For instance

	a	b	c
1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$
2	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$
3	0	$\frac{1}{3}$	$\frac{2}{3}$

is ordinally efficient and envy-free. Here we must invoke weak strategy-proofness in order to characterize the PS assignment. Say that 3 reports $b \succ_3 a \succ_3 c$: after the misreport we are in the other type 5 profile, for which we know that 3 gets $(\frac{2}{3}) b + (\frac{1}{3}) c$. Therefore weak strategyproofness implies $p_{3b} \geqslant \frac{2}{3}$. But 1 does not envy 3 so $\frac{1}{2} + p_{1b} \geqslant p_{3b}$ implying $p_{3b} \leqslant \frac{2}{3}$ and we are done.

For type 6 and $b \succ_3 a \succ_3 c$ we get $p_{3a} = p_{2b} = 0$ from ordinal efficiency and (18). From no envy between 1 and 2 we have $p_{1a} = p_{2a} = \frac{1}{2}$ and from no envy between 1 and 3, we get $\frac{1}{2} + p_{1b} = p_{3b}$ so that $p_{1b} = \frac{1}{4}$ and we are done.

Finally, for type 6 and $b \succ_3 c \succ_3 a$, we have $p_{3a} = p_{2b} = 0$, $p_{1a} = p_{12} = \frac{1}{2}$. If 3 reports $b \succ_3^* a \succ_3^* c$ he or she gets $(\frac{3}{4}) b + (\frac{1}{4}) c$ so $p_{3b} \geqslant \frac{3}{4}$. As 1 does not envy 3, we have $\frac{1}{2} + p_{1b} \geqslant p_{3b} \Leftrightarrow p_{3b} \leqslant \frac{3}{4}$ and we are done.

7. *Proof of Theorem* 2 (*Impossibility Result*)

For $n \geqslant 4$ there does not exist a mechanism which satisfies strategyproofness, equal treatment of equals, and ordinal efficiency. We suppose that there exists such a mechanism $P = P(\succ)$ and will arrive to a contradiction considering the restrictions our three desired properties impose on assignment matrices at several preference profiles.

Note first that it is enough to consider the case $n = 4$. Indeed, for an arbitrary n, look at the following domain restriction. Let agents 1, 2, 3, and 4 prefer objects a_1, a_2, a_3, and a_4 to all others (any ordering among a_1, a_2, a_3, a_4 being admissible), while for $i > 4$ object a_i is the best choice for agent i. O-efficiency implies that for all $i > 4$ object a_i must go to agent i

with probability 1. So the assignment problem is reduced to the first four agents. If there exists a mechanism satisfying the premises of the theorem, its restriction to the above mentioned domain would give such a mechanism for four agents.

In what follows, we will use the following facts.

Fact 1. Suppose that $b \succ_i a$, while $a \succ_j b$ for all $j \neq i$. Then O-efficiency implies $p_{ia} = 0$. (See Step 1 in the proof of Proposition 2.) Also, let $b \succ_i a$ for $i \in I$, while $a \succ_j b$ for $j \notin I$. Then O-efficiency implies $p_{ia} = 0 \ \forall i \in I$ and/or $p_{jb} = 0 \ \forall j \notin I$.

Fact 2. Consider two orderings: $R = a_1 \succ a_2 \succ \cdots \succ a_n$ and $R' = a'_1 \succ a'_2 \succ \cdots \succ a'_n$. Let for some k $\{a_1, ..., a_k\} = \{a'_1, ..., a'_k\}$. Let agent i change his or her preferences from R to R', the preferences of others being the same (R_{-i}). By SP,

$$\sum_{j=1}^{k} p_{ia_j}(R, R_{-i}) = \sum_{j=1}^{k} p_{ia'_j}(R', R_{-i})$$

(of course, the same is true for the sums $j = k+1$ til n).

Fact. 3. Let a be the best object for everyone; then $p_{1a} = p_{2a} = p_{3a} = p_{4a} = \frac{1}{4}$.

Indeed, it is true for an unanimous preference profile by equal treatment of equals. Whenever agent 4 changes his or her preferences, object a being still his or her best, he or she has to receive $p_{4a} = \frac{1}{4}$ by Fact 2, while others have to get $(1 - \frac{1}{4})/3 = \frac{1}{4}$ by equal treatment of equals. If agent 3 changes his or her preferences after that to be like the ordering of agent 4, then he or she still has to receive $p_{3a} = \frac{1}{4}$. Hence by equal treatment of equals $p_{4a} = \frac{1}{4}$, and so again by equal treatment of equals $p_{1,2a} = \frac{1}{4}$. Similar arguments apply when 1, 2 have the same orderings, while 3 and 4 have arbitrary ones (with a as the top)—we just see what happens when 3 changes to be like 4 or 4 changes to be like 3—as well as in the general case.

Fact 4. Let exactly three agents (say, 1, 2, 3) have a as their top choice. Fact 1 gives $p_{4a} = 0$, while Fact 2 implies (as above) that $p_{1a} = p_{2a} = p_{3a} = \frac{1}{3}$.

We proceed by considering several preference profiles. We will write *abcd* instead of $a \succ b \succ c \succ d$, and *abcd*(2) if two agents in the profile have $a \succ b \succ c \succ d$. While we refer to the agent whose preferences are listed first as agent 1, etc., we keep in mind that once we derive some restrictions on P for a given profile, we obtain the same restrictions for all permutations of agents' orderings. Figure 1 records the successive facts we establish for a number of specific profiles.

1)

	a	b	c	d
abcd(3)	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$
badc	0	$\frac{1}{2}$	0	$\frac{1}{2}$

2)

	a	b	c	d
abcd(2)	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	0
abdc(2)	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{2}$

3)

abcd(3)	$\frac{1}{4}$	$\frac{1}{3}$		
adbc	$\frac{1}{4}$	0	0	

4)

abcd(2)	$\frac{1}{3}$			
adbc	$\frac{1}{3}$			
ba··	0	$\frac{7}{12}$		

5)

abcd(3)	$\frac{1}{3}$	$\frac{1}{6}$		
bdac	0	$\frac{1}{2}$	0	

6)

abcd(2)	$\frac{1}{2}$	0	$\frac{1}{2}$	0
badc(2)	0	$\frac{1}{2}$	0	$\frac{1}{2}$

7)

abcd(2)	$\frac{1}{2}$			0
bdac(2)	0	$\boldsymbol{\frac{1}{2}}$	**0**	$\frac{1}{2}$

8)

abcd(2)				
bdac	0	$\frac{1}{2}$	0	
badc	0	$\frac{1}{2}$	0	

9)

abcd(2)	$\frac{1}{3}$			0
bdac	0	$\frac{7}{12}$	$\frac{1}{12}$	$\frac{1}{3}$
adbc	$\frac{1}{3}$	0	0	$\frac{2}{3}$

10)

abcd(2)	$\frac{1}{4}$			0
adbc(2)	$\frac{1}{4}$		$\frac{1}{12}$	

11)

abcd(2)	$\frac{1}{4}$			
adbc	$\frac{1}{4}$	0	0	$\frac{3}{4}$
abdc	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{1}{3}$

FIGURE 1

Profile 1: *abcd*(3), *badc*. By Fact 1, $p_{4a} = p_{4c} = 0$ (objects a, b, then objects c, d); hence $p_{4b} = p_{4c} = \frac{1}{2}$ by Fact 2 (compare with profile "*abcd*(4)": $p_{4a} + p_{4b}$ should not change).

Profile 2: *abcd*(2), *abdc*(2). $p_{ia} = \frac{1}{4}$ by Fact 3 and by a similar argument $p_{ib} = \frac{1}{4}$ for all i. Then it follows from O-efficiency that $p_{1,2d} = p_{3,4c} = 0$. Indeed, $p_{1,2d} > 0$ would imply $p_{3,4c} = 0$ by O-efficiency, which would in turn imply $p_{3,4d} = \frac{1}{2}$ (P is bistochastic!) and hence $p_{1,2d} = 0$.

Profile 3: *abcd*(3), *adbc*. We have that $p_{ia}=\frac{1}{4}$ by Fact 3, while by Fact 1 $p_{4b}=0$ (objects b, d) and $p_{4c}=0$ (objects c, d). Thus $p_{jb}=\frac{1}{3}$ for $j=1, 2, 3$.

Profile 4: *abcd*(2), *adbc*, *bacd* (or *badc*!). By Fact 4, $p_{1a}=p_{2a}=p_{3a}=\frac{1}{3}$, while $p_{4a}=0$. Changing preferences of the last agent to $R'=abcd$ gives us the Profile 3, so by Fact 2 $p_{4a}+p_{4b}$ must be the same as at that profile. Hence, $p_{4b}=\frac{1}{4}+\frac{1}{3}-0=\frac{7}{12}$.

Profile 5: *abcd*(3), *bdac*. $p_{4a}=0$ by Fact 4, while $p_{4b}=\frac{1}{2}$ and $p_{4c}=0$ by Fact 2 (change of preferences of agent 4 to *badc* leads to Profile 1).

Profile 6: *abcd*(2), *badc*(2). When the last agent changes from *badc* to *abcd*, we obtain Profile 1. Thus $p_{4a}+p_{4b}=\frac{1}{2}$, as at Profile 1. Using equal treatment of equals we get $p_{ia}+p_{ib}=p_{ic}+p_{ib}=\frac{1}{2}\ \forall i$. Finally, using the same argument as for Profile 2, it follows from O-efficiency that $p_{1,2b}=p_{3,4a}=0$, etc.

Profile 7: *abcd*(2), *bdac*(2). O-efficiency implies that in each pair $(p_{3,4a}, p_{1,2b})$, $(p_{3,4a}, p_{1,2d})$, and $(p_{3,4c}, p_{1,2d})$ at least one probability must be zero. Suppose $p_{3,4a}>0$; then $p_{1,2b}=p_{1,2d}=0$. Hence (P is bistochastic!) $p_{3,4b}=p_{3,4d}=\frac{1}{2}$ and so $p_{3,4a}=0$. Thus $p_{3,4a}=0$. By similar argument, $p_{1,2d}=0$, and so $p_{1,2a}=p_{3,4d}=\frac{1}{2}$. Before completing the assignment matrix for this profile we need to consider another profile, namely:

Profile 8: *abcd*(2), *bdac*, *badc*. $p_{3a}=0$ for Fact 1 (objects a, d); $p_{3b}=\frac{1}{2}$ and $p_{3c}=0$ by Fact 2 (compare with Profile 6, when agent 3 changes from *bdac* to *badc*). Look now at the last agent with preferences *badc*. When he or she changes to *abcd* we obtain Profile 5. Thus $p_{4a}+p_{4b}=\frac{1}{3}+\frac{1}{6}=\frac{1}{2}$. When he or she changes to *bdac*, we obtain Profile 7. Thus p_{4b}, p_{4c} must be the same as at Profile 7. So $p_{4b}+p_{4c}=1-(0+\frac{1}{2})=\frac{1}{2}$. Suppose that $p_{4a}>0$; then by O-efficiency $p_{1,2b}=0$, and so $p_{4b}=\frac{1}{2}$; i.e., $p_{4a}=0$. Thus we have $p_{4a}=0=p_{4c}$, and $p_{4b}=\frac{1}{2}$.

Return to Profile 7. When the last agent changes from *bdac* to *badc*, we obtain Profile 8. So by Fact 2 p_{4b} does not change; i.e., at the Profile 7 $p_{4b}=\frac{1}{2}$, $p_{4c}=0$ (see the bold numbers in the matrix of Fig. 1).

Profile 9: *abcd*(2), *bdac*, *adbc*. By Fact 4, $p_{1,2,4a}=\frac{1}{3}$, $p_{3a}=0$. By Fact 1 (objects b, d) $p_{4b}=0$. By Fact 2 (agent 3 changes from *bdac* to *ba*.. and we get Profile 4), $p_{3b}=\frac{7}{12}$. Also by Fact 2 (agent 4 changes from *adbc* to *bdac* and we get Profile 7), $p_{4c}=0$. Hence, $p_{4d}=\frac{2}{3}$. Next, $p_{1d}+p_{2d}+p_{3d}=1-p_{4d}=\frac{1}{3}$, so $p_{3c}=1-\frac{7}{12}-p_{3d}\geqslant 1-\frac{7}{12}-\frac{1}{3}=\frac{1}{12}>0$; hence by O-efficiency $p_{1,2d}=0$.

Profile 10: *abcd*(2), *adbc*(2). $p_{ia}=\frac{1}{4}$ for all i by Fact 3. When agent 3 changes from *adbc* to *bdac*, we get Profile 9. Thus by Fact 2 $p_{3c}=\frac{1}{12}\,(=p_{4c})>0$. Then by O-efficiency $p_{1,2d}=0$.

Profile 11: *abcd*(2), *adbc*, *abdc*. By Fact 3, $p_{ia} = \frac{1}{4}$ for all i. Consider agent 4 with preferences *abdc*. When he or she changes to *adbc*, we get Profile 10; so by Fact 2 $p_{4c} = \frac{1}{12}$. When he or she changes to *abcd*, we get Profile 3, so by Fact 2 $p_{4b} = \frac{1}{3}$. Hence, $p_{4d} = \frac{1}{3}$. Consider now agent 3 with preferences *adbc*. By Fact 1 (objects b, d), $p_{3b} = 0$. If agent 3 changes to *abdc*, we get Profile 2. So by Fact 2 $p_{3c} = 0$; hence $p_{3d} = \frac{3}{4}$. But then $p_{3d} + p_{4d} = \frac{3}{4} + \frac{1}{3} > 1$, which is the desired contradiction.

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